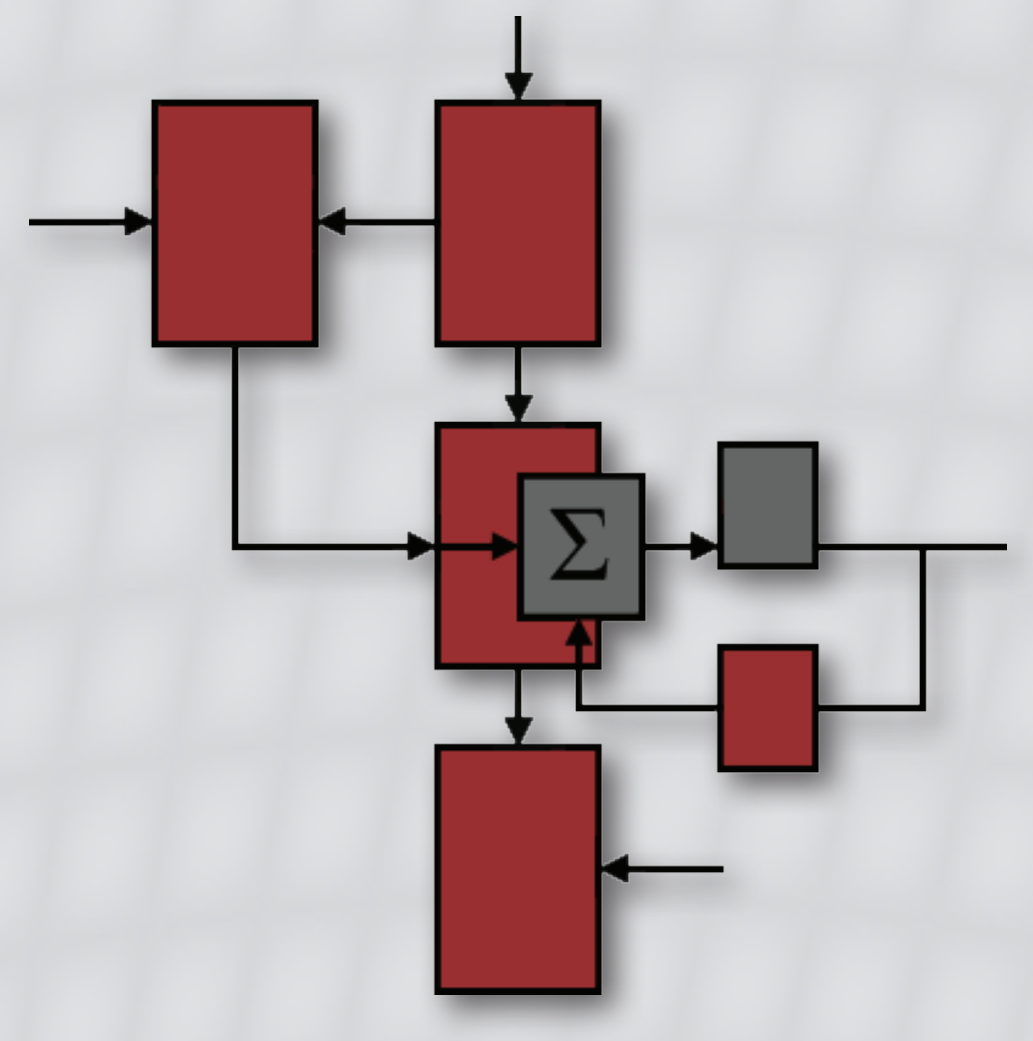

Minimum Mean Bayes Risk Error Quantization of Prior Probabilities

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Problem Overview

Hypothesis Testing Scenario

- Object in one of M states: $\{h_0,\dots,h_{M-1}\}$ with $\Pr[H=h_m]=p_m$
- Prior probability vector $\boldsymbol{p}=\begin{bmatrix}p_0 & \cdots & p_{M-1}\end{bmatrix}^T$
- Population of objects — each object's prior probability drawn from $f_{\boldsymbol{P}}(\boldsymbol{p})$
- Noisy measurement Y
- Task: for a given object with prior probability $\boldsymbol{p}$, estimate h from y under Bayes risk objective with costs c_{ij}
- Constraint: $\boldsymbol{P}$ is quantized
- Goal: design best K-point quantizer to minimize average Bayes risk error over the population $f_{\boldsymbol{P}}(\boldsymbol{p})$

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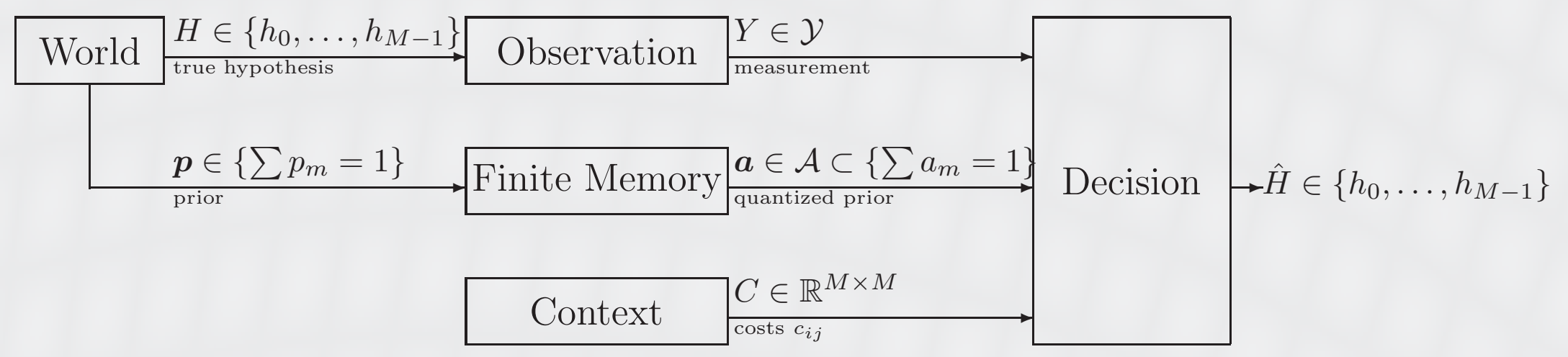
Block Diagram



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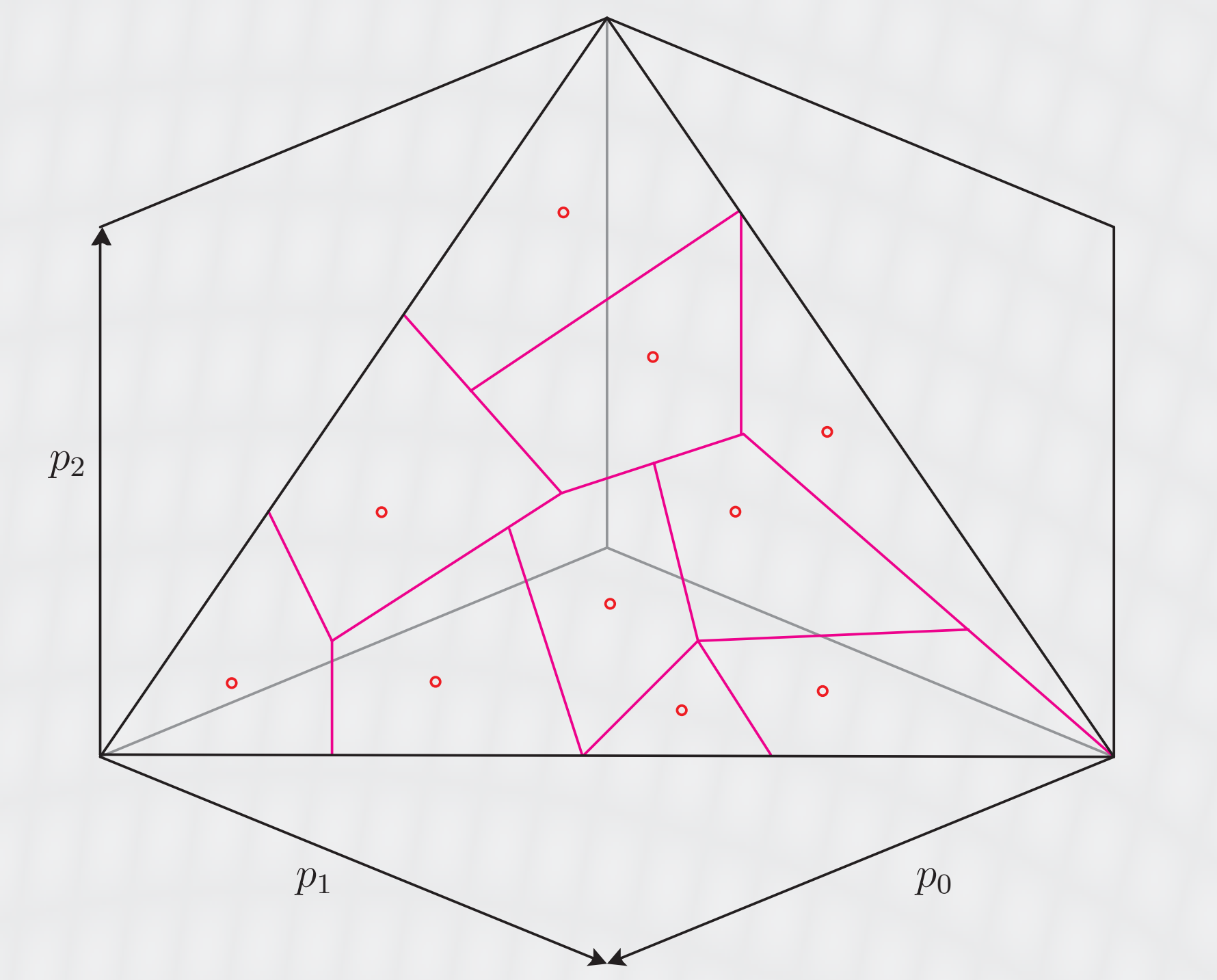
A Quantizer over the Population $f_{\boldsymbol{P}}(\boldsymbol{p})$, $M=3$



Background

Binary Bayesian Hypothesis Testing

- Hypotheses h_0, h_1, and prior probabilities p_0, p_1, such that $p_0=\Pr[H=h_0]$ and $p_1=\Pr[H=h_1]=1-p_0$
- Noisy measurement Y with likelihoods $f_{Y|H}(y|h_0)$ and $f_{Y|H}(y|h_1)$
- Function $\hat{h}(y)$ maps every possible y to either h_0 or h_1
- $\hat{h}(\cdot)=\arg\min_{f(\cdot)} E\left[c\left(H,f(Y)\right)\right]$ with solution $\frac{f_{Y|H}(y|h_1)}{f_{Y|H}(y|h_0)} \underset{\hat{h}(y)=h_0}{\overset{\hat{h}(y)=h_1}{\gtrless}} \frac{p_0(c_{10}-c_{00})}{p_1(c_{01}-c_{11})}$, where $c_{ij}=c(h_i,h_j)$
- Errors: $p_E^{\mathrm{I}}=\Pr[\hat{h}(Y)=h_1|H=h_0]$ and $p_E^{\mathrm{II}}=\Pr[\hat{h}(Y)=h_0|H=h_1]$
- Bayes risk: $J=(c_{10}-c_{00})p_0p_E^{\mathrm{I}}+(c_{01}-c_{11})p_1p_E^{\mathrm{II}}+c_{00}p_0+c_{11}p_1$

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Scalar Quantization

- K-point quantizer $v_K(\cdot)$ for $f_{P_0}(p_0)$, $p_0\in[0,1]$ partitions the domain into intervals $\mathcal{R}_1=[0,b_1]$, $\mathcal{R}_2=(b_1,b_2]$, $\mathcal{R}_3=(b_2,b_3]$, $\dots$, $\mathcal{R}_K=(b_{K-1},1]$
- $v_K(p_0)=a_k$ for $p_0\in\mathcal{R}_k$
- Design $v_K(\cdot)$ to minimize $D=E[d(P_0,v_K(P_0))]=\int d(p_0,v_K(p_0))f_{P_0}(p_0)dp_0$
- Conditions for quantizer optimality: nearest neighbor condition, centroid condition, zero probability of boundary condition [Gersho & Gray, 1992]

Bayes Risk Error Distortion

Bayes Risk Function Properties

- Assuming $M=2$ and $c_{00}=c_{11}=0$, Bayes risk as a function of p_0 is $J(p_0)=c_{10}p_0p_E^{\mathrm{I}}(p_0)+c_{01}(1-p_0)p_E^{\mathrm{II}}(p_0)$
- $J(p_0)$ is zero at $p_0=0$ and $p_0=1$
- $J(p_0)$ is positive, concave, and continuous in the interval $(0,1)$

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Mismatched Bayes Risk

- If the true prior probability is p_0, but $\hat{h}(y)$ is designed using some other value a, there is mismatch
- Mismatched Bayes risk function $\tilde{J}(p_0,a)=c_{10}p_0p_E^{\mathrm{I}}(a)+c_{01}(1-p_0)p_E^{\mathrm{II}}(a)$
- $\tilde{J}(p_0,a)$ is a linear function of p_0 with slope $(c_{10}p_E^{\mathrm{I}}(a)-c_{01}p_E^{\mathrm{II}}(a))$ and intercept $c_{01}p_E^{\mathrm{II}}(a)$
- $\tilde{J}(p_0,a)$ is tangent to $J(p_0)$ at $p_0=a$
- $\tilde{J}(p_0,a)\geq J(p_0)\geq 0$

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Bayes Risk Error

- Define distortion function for quantization
- Bayes risk error: $d(p_0,a)=\tilde{J}(p_0,a)-J(p_0)$
- Nonnegative and only equal to zero when $p_0=a$
- Convex and continuous in p_0 for all a

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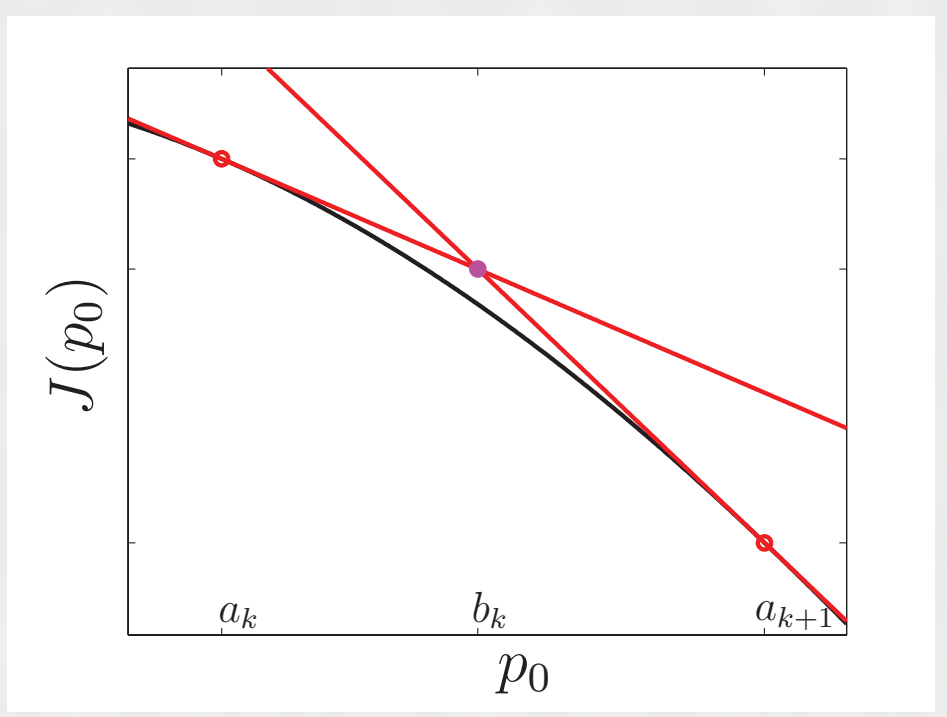
Comparison to Previous Work

- Previous work combining detection and quantization quantizes y, not p_0 [Kassam, 1977], [Poor & Thomas, 1977], [Gupta & Hero, 2003]
- Also only approximates Bayes risk function instead of using it directly

Quantizer Optimality Conditions

Nearest Neighbor Condition

- For fixed representation points $\{a_k\}$, derive optimal boundary points $\{b_k\}$
- Given $p_0\in[a_k,a_{k+1}]$, if $\tilde{J}(p_0,a_k)<\tilde{J}(p_0,a_{k+1})$, then the Bayes risk error is minimized when p_0 represented by a_k and vice versa
- $b_k\in[a_k,a_{k+1}]$ is abscissa of intersection between $\tilde{J}(p_0,a_k)$ and $\tilde{J}(p_0,a_{k+1})$



- $b_k=\frac{c_{01}\left(p_E^{\mathrm{II}}(a_{k+1})-p_E^{\mathrm{II}}(a_k)\right)}{c_{01}\left(p_E^{\mathrm{II}}(a_{k+1})-p_E^{\mathrm{II}}(a_k)\right)-c_{10}\left(p_E^{\mathrm{I}}(a_{k+1})-p_E^{\mathrm{I}}(a_k)\right)}$

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Centroid Condition

- For fixed boundary points $\{b_k\}$, derive optimal representation points $\{a_k\}$
- Minimize $D=\sum_{k=1}^{K}\int_{\mathcal{R}_k}\left(\tilde{J}(p_0,a_k)-J(p_0)\right)f_{P_0}(p_0)dp_0$ for each interval separately
- Let $I_k^{\mathrm{I}}=\int_{\mathcal{R}_k}p_0f_{P_0}(p_0)dp_0$ and $I_k^{\mathrm{II}}=\int_{\mathcal{R}_k}(1-p_0)f_{P_0}(p_0)dp_0$
- $a_k=\arg\min_a\left\{c_{10}I_k^{\mathrm{I}}p_E^{\mathrm{I}}(a)+c_{01}I_k^{\mathrm{II}}p_E^{\mathrm{II}}(a)\right\}$
- Uniquely minimize by setting derivative equal to zero since $d(p_0,a)$ has exactly one stationary point
- a_k is solution to $c_{10}I_k^{\mathrm{I}}\frac{dp_E^{\mathrm{I}}(a_k)}{da_k}+c_{01}I_k^{\mathrm{II}}\frac{dp_E^{\mathrm{II}}(a_k)}{da_k}=0$

Gaussian Examples

Signal and Noise Model

- Model: $Y=s_m+W$, where $s_0=0$, $s_1=\mu$, and W is a zero-mean Gaussian random variable with variance σ^2
- The likelihood functions are $f_{Y|H}(y|h_0)=\mathcal{N}(y;0,\sigma^2)=\frac{1}{\sigma\sqrt{2\pi}}e^{-y^2/2\sigma^2}$ and $f_{Y|H}(y|h_1)=\mathcal{N}(y;\mu,\sigma^2)=\frac{1}{\sigma\sqrt{2\pi}}e^{-(y-\mu)^2/2\sigma^2}$

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Optimality Conditions

- The two error probabilities are $p_E^{\mathrm{I}}(p_0)=Q\left(\frac{\mu}{2\sigma}+\frac{\sigma}{\mu}\ln\left(\frac{c_{10}p_0}{c_{01}(1-p_0)}\right)\right)$ and $p_E^{\mathrm{II}}(p_0)=Q\left(\frac{\mu}{2\sigma}-\frac{\sigma}{\mu}\ln\left(\frac{c_{10}p_0}{c_{01}(1-p_0)}\right)\right)$, where $Q(\alpha)=\frac{1}{\sqrt{2\pi}}\int_\alpha^\infty e^{-x^2/2}dx$
- Substitute error probabilities into nearest neighbor condition expression
- Derivatives are $\left.\frac{dp_E^{\mathrm{I}}(p_0)}{dp_0}\right|_{p_0=a_k}=-\frac{1}{\sqrt{2\pi}}\frac{\sigma}{\mu}\frac{1}{a_k(1-a_k)}e^{-\frac{1}{2}\left(\frac{\mu}{2\sigma}+\frac{\sigma}{\mu}\ln\left(\frac{c_{10}a_k}{c_{01}(1-a_k)}\right)\right)^2}$ and $\left.\frac{dp_E^{\mathrm{II}}(p_0)}{dp_0}\right|_{p_0=a_k}=+\frac{1}{\sqrt{2\pi}}\frac{\sigma}{\mu}\frac{1}{a_k(1-a_k)}e^{-\frac{1}{2}\left(\frac{\mu}{2\sigma}-\frac{\sigma}{\mu}\ln\left(\frac{c_{10}a_k}{c_{01}(1-a_k)}\right)\right)^2}$
- Substitute derivatives into centroid condition expression and solve to obtain $a_k=\frac{I_k^{\mathrm{I}}}{I_k^{\mathrm{I}}+I_k^{\mathrm{II}}}$

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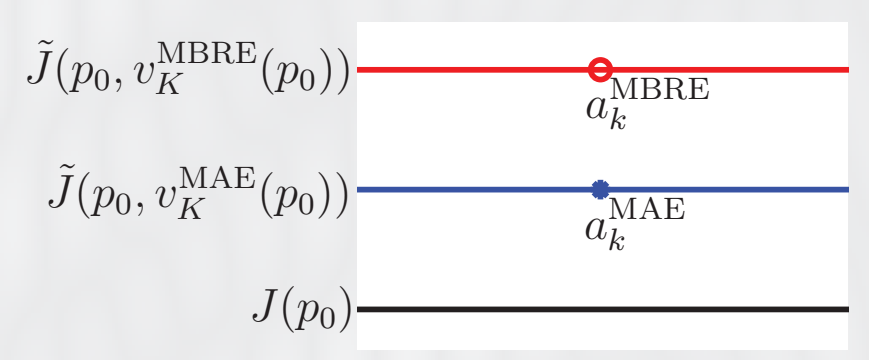
Setup of Examples

- Minimum mean Bayes risk error (MBRE) quantizer designed for $\mu=1$ and $\sigma=1$ using Lloyd-Max algorithm
- Compared to minimum mean absolute error (MAE) quantizer ($d(p_0,a)=|p_0-a|$)
- Plots show $f_{P_0}(p_0)$; MBRE as a function of K for both quantizers; and $\tilde{J}(p_0,v_K(p_0))$ for four values of K for both quantizers

Legend



In mean Bayes risk error plot, red line is for MBRE-optimal quantizer and blue line is for MAE-optimal quantizer.

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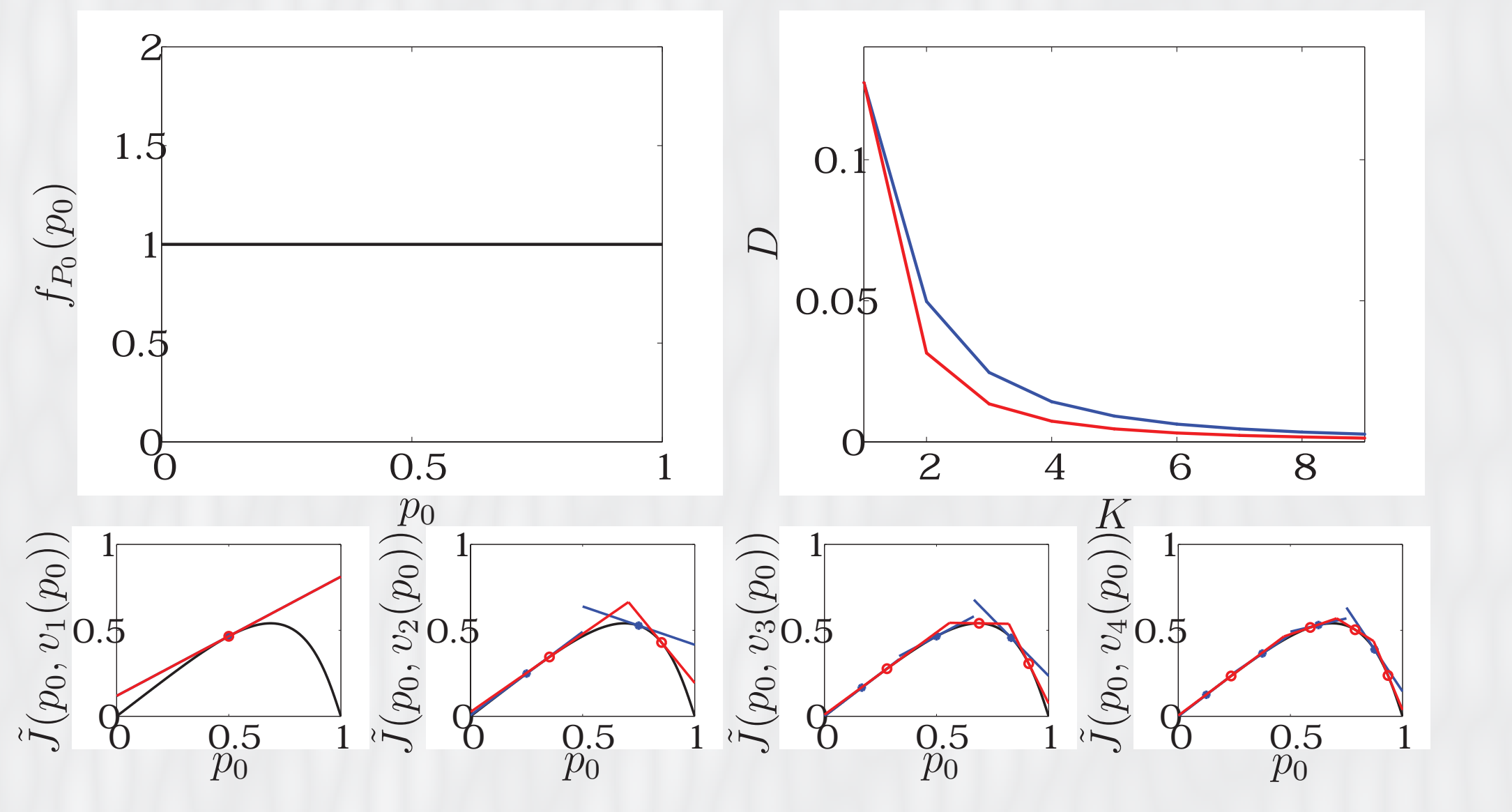
$f_{P_0}(p_0)$ ~ Uniform; $c_{10}=1$, $c_{01}=4$



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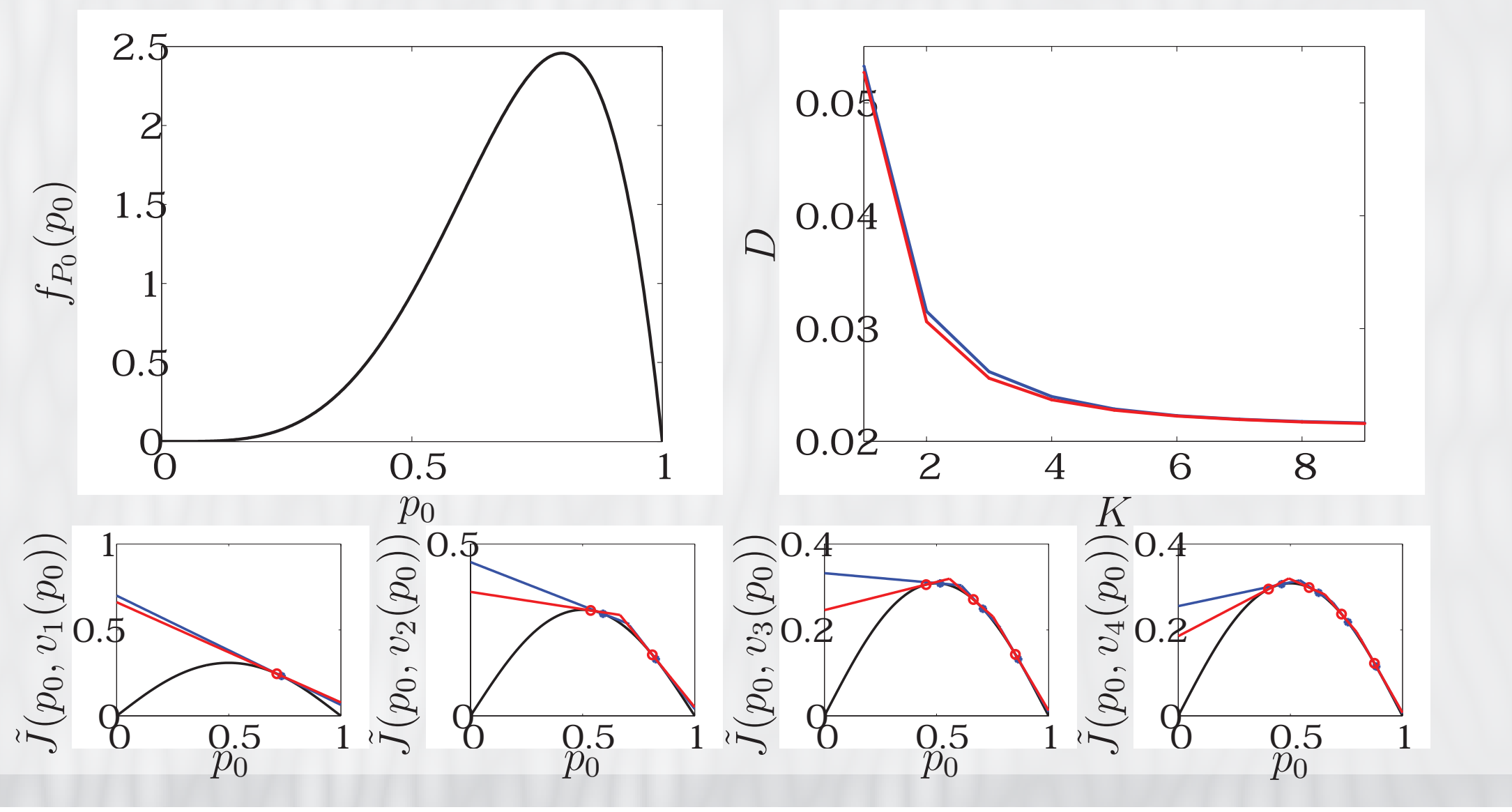
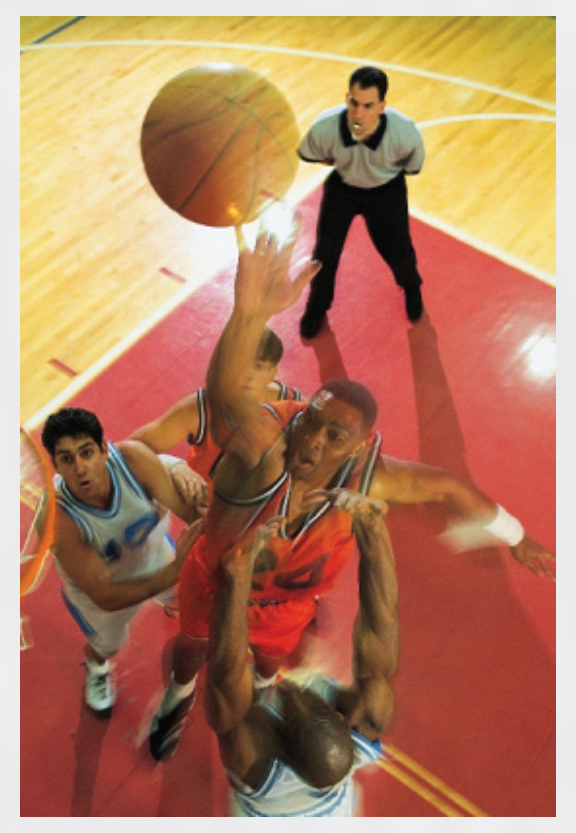
$f_{P_0}(p_0)$ ~ Beta(5,2); $c_{10}=1$, $c_{01}=1$



Human Decision Making

Human Decision Makers



- Limited information processing capacity
- Handle 7 ± 2 categories without getting confused [Miller, 1956]
- Assumption: perform optimal Bayesian hypothesis testing with quantized priors
- Essentially automatic race and gender categorization, particularly when lacking the time, motivation, or cognitive capacity to think deeply [Macrae & Bodenhausen, 2000]
- Assumption: separate quantizers for different races and genders with total point constraint K_t

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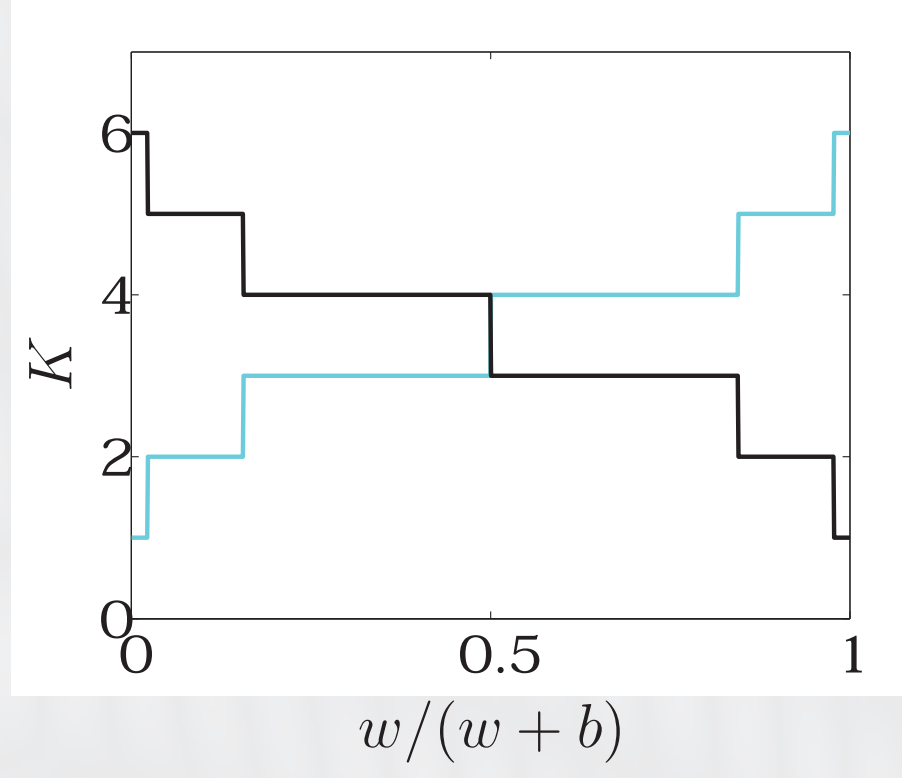
Two Population Quantization

- Two identical populations B and W with common $f_{P_0}(p_0)$
- Separate quantizers $v_{K_b}(\cdot)$ and $v_{K_w}(\cdot)$ with total constraint $K_t=K_w+K_b$
- Exposure of decision maker to populations not identical: b and w
- Extend MBRE to two population case: $D^{(2)}=\frac{w}{w+b}E[\tilde{J}(P_0,v_{K_w}(P_0))]+\frac{b}{w+b}E[\tilde{J}(P_0,v_{K_b}(P_0))]-E[J(P_0)]$
- Optimization involves point allocation and optimal separate quantizer



- Optimal point allocation as a function of $w/(w+b)$ for $f_{P_0}(p_0)$ ~ Beta(5,2), $c_{10}=c_{01}=1$, and $K_t=7$
- K_b = black line, K_w = cyan line
- Better Bayes risk performance for population W when $K_b<K_w$ and vice versa
- Predictions from model consistent with face recognition experiments [Meissner & Brigham, 2001]

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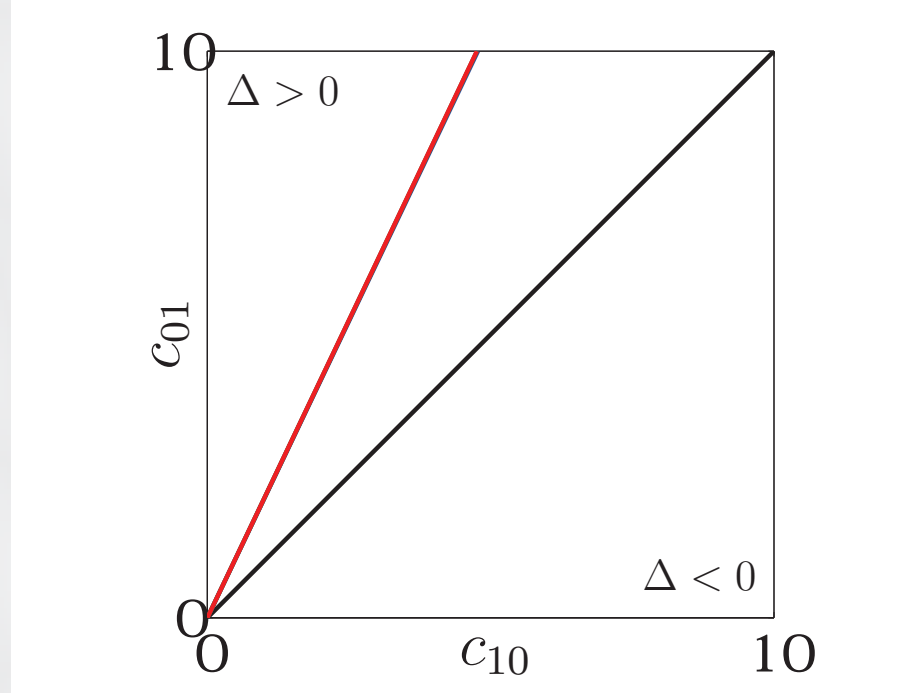
Effect of Bayes Costs

- Bayes risk performance includes both Type I and Type II errors
- Often, only detection rate is observable experimentally: $\Pr[\tilde{h}_K(Y)=h_1]=1-p_0+p_0p_E^{\mathrm{I}}(v_K(p_0))-(1-p_0)p_E^{\mathrm{II}}(v_K(p_0))$
- If decision maker is member of W with $w>b$ and $\Delta>0$, where $\Delta=E\left[\Pr[\tilde{h}_{K_b}(Y)=h_1]-\Pr[\tilde{h}_{K_w}(Y)=h_1]\right]$, then out-of-population bias; if $\Delta<0$, then in-population bias



- Δ is a function of c_{10} and c_{01}
- Dividing line between in-population bias and out-of-population bias is plotted for Uniform (black) and Beta(5,2) (colored) distributions
- Precautionary principle, $c_{10}\gg c_{01}$, yields out-of-population bias
- Pushful principle, $c_{01}\gg c_{10}$, yields in-population bias
- Empirical evidence of human decision making shows $\Delta>0$ [Donohue & Levitt, 2001], [Price & Wolfers, 2007] — consistent with precautionary principle

Conclusion

Memory Constraints Limit Decision Making

- Formulated hypothesis testing under memory constraints rather than communication constraints
- Derived optimal quantizer of prior probability distribution under Bayes risk objective and determined performance
- Quantized prior hypothesis testing combined with theories of social cognition and empirical facts about segregation leads to generative model of discrimination
- Decision making biased despite identical population distributions and no malicious intent by decision maker

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