Elastic Shape Models for Face Analysis Using Curvilinear Coordinates

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Abstract   This paper studies the problem of analyzing variability in shapes of facial surfaces using a Riemannian framework, a fundamental approach that allows for joint matchings, comparisons, and deformations of faces under a chosen metric. The starting point is to impose a curvilinear coordinate system, named the Darcyan coordinate system, on facial surfaces; it is based on the level curves of the surface distance function measured from the tip of the nose. Each facial surface is now represented as an indexed collection of these level curves. The task of finding optimal deformations, or geodesic paths, between facial surfaces reduces to that of finding geodesics between level curves, which is accomplished using the theory of elastic shape analysis of 3D curves. Elastic framework allows for nonlinear matching between curves and between points across curves. The resulting geodesics provide optimal elastic deformations between faces and an elastic metric for comparing facial shapes. We demonstrate this idea using examples from FSU face database.

1 Introduction

There is an increasing interest in analyzing shapes of facial surfaces with many applications including biometrics, facial surgery, video communications, and 3D animation. This interest is fuelled by the advent of cheaper and lighter scanners that can provide high-resolution measurements of both geometry and texture of human facial surfaces. One general goal here is to develop computational tools for analyzing 3D face data. In particular, one is interested in comparing the shapes of facial surfaces. Such a tool can be used to recognize human beings according to their facial shapes, to measure changes in a facial shape due to a surgery, or to study/capture the variations in facial shapes during conversations and expressions of emotions. Additionally, a subproblem may be to find an optimal deformation of one surface into another, under some chosen criterion. These deformations can be useful in defining summary statistics of a collection of faces. For example, one can compute an *average* of faces of different people, under different facial expressions. Efficient tools for understanding and studying variability in facial shapes are of great importance in our biometric-oriented society. Accordingly, the *main theme of this paper is to develop a computational framework for analyzing shapes of facial surfaces.*

1.1 Major Issues in Surface Analysis

What are the difficulties in performing standard shape analysis on the face data? The main issue is that there is no canonical coordinate system to represent and study the geometry of a nonlinear surface, such as a face. To highlight this issue, contrast the geometry of a surface with that of a curve. The points along a curve have a fixed ordering which allows us to impose a coordinate system for studying shapes of curves. Although the parametrization for a curve may not be unique, the task of analyzing shapes of curves modulo all possible parameterizations is still tractable. More specifically, the space of all possible parameterizations

of a curve is the set of all one-dimensional diffeomorphisms, and there exist efficient algorithms (such as the dynamic programming algorithm) for optimally matching curves over all parameterizations. For surfaces, there is no natural ordering of points and, thus, no natural way of parameterizing them. Thus, the goal of analyzing shapes of surfaces, modulo all possible re-parameterizations, becomes quite difficult and leads to enormous computational challenges.

1.2 Past Work

Past literature has tackled this fundamental issue with a varying degree of success, depending on the ultimate application. If we divide the main tasks in facial shape analysis into **representing**, **matching**, **comparing**, and **deforming**, several past papers that have addressed these individual tasks.

1. **Surface Representations & Feature Comparisons**: For comparing faces, a common theme in this literature has been to represent facial surfaces by certain geometrical features, such as the height function [16], convex regions, areas with high curvatures [15], saddle points, etc. These methods provide reasonable results in classification and clustering of faces, but they do not study deformations nor develop statistical analysis of shapes. Additionally, although some of these features are intuitively meaningful, their computations, e.g. those involving second derivatives, may be susceptible to observation noise. Other approaches, such as those based on shape distribution [33] and conformal geometry [40], have also been proposed for matching surfaces. Beumier and Acheroy [2] have used shapes of facial profiles to compare faces.
   Another commonly used approach, that relates to our paper, is to define smooth functions, more specifically Morse functions, on surfaces and to use the level curves of these functions for defining coordinates on surfaces. Samir et al. [34] used the level curves of the height function to define and study shapes of facial curves. Bronstein et al. [4] used the geodesic distances on facial surfaces to perform surface matching and comparisons. This distance has also been used by many other authors, including Gupta [14] and Samir et al. [36], etc. Mpiperis et al. [32] have used this representation to match features across faces.
2. **Matching Facial Surfaces**: There are two types of matching algorithms: rigid and non-rigid. Rigid surface matching, often viewed as a matching of two point clouds in $\mathbb{R}^3$, is performed using the Iterative Closest Point (ICP) algorithm (see for example [1]) or its modifications. For non-rigid matching, several ideas have been proposed. Lu et al. [26,27] have used thin plate splines to perform non-rigid matching of faces. Chui and Rangarajan [8] have presented a robust algorithm for non-rigid deformation and registration of point sets. Chang et al. [6] have suggested restricting to some prominent parts of faces, such as the nose region, for performing the matching. One idea that is often used in medical image analysis is the use of mutual information to register points across high-dimensional images. This idea has been applied to matching of facial surfaces by Wang et al. [41]. However, this approach is often applied to matching using statistics of pixel values in images rather than geometric properties of a surface.
   Bronstein et al. [3] study the problem of matching points across surfaces in an isometric manner. Here the task of matching surfaces has been accomplished by minimizing the Gromov-Hausdorff distance using multidimensional scaling [28]. A Riemannian analysis of facial surfaces developed here is different from Bonstein et al. in that the matching and deformation energies are based on lengths of geodesic paths (under the chosen Riemannian metric) and not a global distance like Gromov-Hausdorff. For shape analysis of planar shapes using Hausdorff distances, Charpiat et al. [7] have presented a Riemannian framework, but such a connection for surfaces remains to be presented.
3. **Deforming Facial Surfaces**: In recent years there has been focus on deforming surfaces, one into another, under a chosen criterion. Grenander's *deformable template* theory [11] has been successfully applied to studying shapes of anatomical parts using medical images [30,12]. The set of non-rigid deformations can be subdivided into linear and nonlinear deformations. Nonlinear deformations imply local stretching, compression, and bending of surfaces to match each other, and are also referred to as **elastic** deformations. Earlier attempts at elastic matching utilized graphs based on texture images of faces [42,24]. Kakadiaris et al. [19] utilize an *annotated face model* to study geometrical variability across faces. Glaunes et al. [9] have studied diffeomorphic matching of points sets in two and three-dimensions. Litke et al. [25] take an interesting approach by representing surfaces as 2D images, using arbitrary parameterizations, and matching of surfaces is accomplished by matching the corresponding 2D image coordinates. The

cost function for matching includes distortions introduced due to parameterizations, deformations, and features mismatch. A recent paper by Kilian et al. [21] describes a numerical approach to computing geodesic paths between registered triangulated surfaces using Euclidean metrics; the invariance of this approach to arbitrary parameterizations of surfaces was not discussed.

1.3 Our Approach

We present a unified Riemannian framework that provides optimal matching, comparisons and deformations of faces using a single elastic metric. The following items highlight the contribution of this paper.

1. **Darcyan Curvilinear Coordinate System**: Rather than allowing arbitrary parameterizations of facial surfaces, as in [25], we choose a specific coordinate system that is natural for analyzing shapes of facial surfaces. We use the geodesic distance function from a fixed reference point to define this system. Being intrinsic to the surface, it is invariant to rigid motions (rotations and translations). The fact that it deforms with the changes in facial expressions provides a possibility of countering variability in facial shapes induced by changes in facial expressions, as noted by several earlier papers [4,14]. We will denote the coordinate system by Ξ and points in Ξ by ξ. *The level sets of coordinates ξ_1 or ξ_2 are curves that vary in shapes from face to face.* This fact is exploited in this paper to characterize shapes of facial surfaces. Such a curvilinear coordinate system has been proposed earlier for use in medical image analysis by Grenander et al. [13] who referred to it as the **Darcyan** coordinate system, after the famous shape analyst D'Arcy Thompson.
2. **Elastic Matching**: An important requirement of a face analysis system is the non-rigid or elastic matching of points across faces. We claim that the task of elastic matching becomes relatively simpler in a Darcyan coordinate system. Let Ξ be the Darcyan coordinate system associated with one face and Ξ' with another. The main task in shape matching is to find a mapping $\phi : \Xi \mapsto \Xi'$ that is a diffeomorphism (smooth and invertible, with a smooth inverse). If ϕs are considered random, then shapes should be compared by integrating over all possible ϕs.

$$d(S_1, S_2) = \int_\phi d_\kappa(\kappa^1(\xi), \kappa^2(\phi(\xi))) P(\phi) d\phi \ ,$$

where
   - κ is some geometric descriptor being used to compare the surfaces,
   - d_κ is an appropriate distance on the space of that descriptor, and
   - $P(\phi)$ is the probability density on ϕ.

However, in practice one uses an optimal ϕ under a pre-determined criterion, to match shapes. This requires solving an optimization problem over the set of all mapping ϕs.

$$d(S_1, S_2) \approx d_\kappa(\kappa^1(\xi), \kappa^2(\phi^*(\xi))), \quad \text{where} \quad \phi^* = \underset{\phi}{\operatorname{argmin}}\, d_\kappa(\kappa^1(\xi), \kappa^2(\phi(\xi))) \ .$$

The advantage of a Darcyan coordinate system is that the search for this optimal ϕ can be constrained further. While in past papers, e.g. Litke et al. [25], surface matching has been studied as a full 2D matching problem, in our approach it is restricted to two 1D matching problems. Here, the mapping $\phi(\xi_1, \xi_2)$ decomposes into the form $\phi_1(\xi_1)\phi_2(\xi_2; \xi_1)$. The first function ϕ_1 simply matches the level curves of ξ_1 on S_1 with the level curves of ξ_1' on S_2. Then, the second function ϕ_2 matches points across the curves previously matched using ϕ_1. This function (ϕ_2) can be different for different values of ξ_1 and, hence, our notation shows an explicit dependence of ϕ_2 on ξ_1. The search for optimal ϕ_1 and ϕ_2 is performed using the same elastic energy function and the computational implementation is based on multiple uses of the dynamic programming algorithm.
3. **Optimal Deformation**: The process of matching points and curves across faces also results in the optimal elastic deformation between faces. As a basic unit for this deformation, we obtain geodesic paths between corresponding (matched) curves from one face to the other; these paths provide optimal elastic deformations between those curves. Each face is represented as an indexed collection of curves and, thus, a composition of these individual geodesics provides a global geodesic (and an optimal elastic deformation) between the two faces. A major advantage of a Riemannian framework is that it provides geodesic paths,

between faces, that are necessary for a full statistical analysis. This is often not feasible in case of global distance based matching algorithms, e.g. in case of Gromov-Hausdorff distance [3].

The main differences between this paper and our earlier papers are as follows:

1. In Samir et al. [35] and [34], we used the level curves of the height function to define facial curves. Since these curves are planar, we used shape analysis of planar curves [23] to compare and deform faces; nonlinear matching problem was not studied here (that is, the mapping was fixed to be linear). The metric used was a non-elastic metric and the shooting method was used for computing geodesics between planar curves. Consequently, this representation is not fully invariant to rigid motions of the face.
2. In Samir et al. [36], we used the level curves of the geodesic distance function that resulted in 3D curves. We used a non-elastic metric and a path-straightening method to compute geodesics between these curves [22]. Here also, the matching was not studied and the correspondence of curves and points across faces was simply linear.
3. In the current paper, we extend the earlier Riemannian framework to be fully elastic by defining a formal coordinate system on facial surfaces and using that to solve the matching, deformation, and comparison problems. Two types of elastic matchings are developed – one for matching the curves and one for matching points on the curves. Curves are represented using a square-root velocity function that has recently been introduced for studying shapes of elastic curves [18,17].

The rest of this paper is organized as follows. We introduce the Darcyan coordinate system (ξ_1, ξ_2) on facial surfaces in Section 2. In Section 3, we summarize ideas from [18] on elastic matching of points on closed curves in $\mathbb{R}^3$ and the resulting deformations of level curves of ξ_1, while in Section 4, we use ideas from dynamic programming to match curves on facial surfaces and to deform facial surfaces. In Section 5, we present some examples on elastic deformation facial surfaces and some their use for applications in face classification. We finish the paper with a summary in Section 6.

2 Curvilinear Coordinate System

The starting point of our framework is to define an intrinsic coordinate system on each facial surface that is natural for studying shape variability. This system is used to decompose elastic matching/deformation of facial surfaces into simpler, more elementary components. In this system the level curves are neither straight lines nor circles; these are curves that lie completely on the surface and locally match the geometry of the surface. Let S be a facial surface assumed to be a two-dimensional differentiable manifold that contains its boundaries. This is obtained in practice by filling the holes in the data obtained from 3D scanners, and by smoothly interpolating the resulting mesh to proper dimensionality of all patches on the surface. Furthermore, S is endowed with a Riemannian structure using the Euclidean metric in $\mathbb{R}^2$, the tangent space at any interior point of S. Let $r \in S$ denote the tip of the nose on S.

Proposition 1 *For any point p, there exists a shortest geodesic, connecting r and p, that lies completely in S. The shortest geodesic may not be unique, but there exists at least one.*

The proof comes from the fact that S is a finite-dimensional, complete, and differentiable manifold. Based on this proposition, one can define a smooth function on S using geodesic distances as follows:

Definition 1 *For any point $p \in S$, define a function $f : S \mapsto \mathbb{R}$ such that $f(p)$ is the length of the shortest geodesic from r to p.*

Since there exists a shortest geodesic from r to p, for all p, the function f is well defined. Also, since S is a differentiable manifold, the function f is a continuous function.

Remark: We caution the reader that this use of geodesic on facial surfaces is different from the later construction of geodesic paths between two curves or two surfaces under appropriate Riemannian metrics. Here, we compute geodesics on S, a two-dimensional manifold with boundaries, to define f, while later on we will compute geodesics on infinite-dimensional spaces of curves and surfaces to realize optimal deformations between their elements.
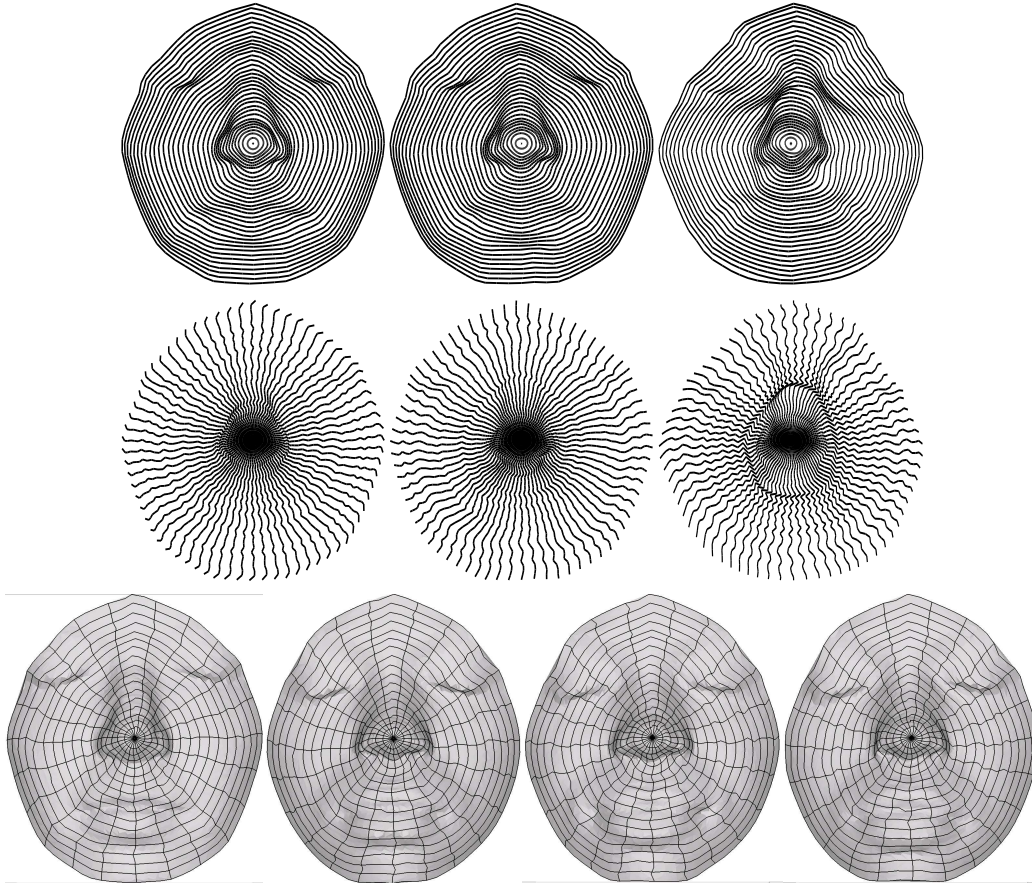
Fig. 1 Top row: Level curves of ξ_1 and middle row: level curves of ξ_2 for three different facial surfaces. Bottom row: full Darcyan coordinate system on some faces.

Define a coordinate system on S using f as follows. Define the level curves of f as the level curves of the first coordinate ξ_1, the value of ξ_1 being equal to the value of f. Shown in the top row of Figure 1 are examples of these curves for three different faces. Next, use the streamlines of ∇f, starting from r, as the level curves of the second coordinate ξ_2. The value of ξ_2 along such curves will be dependent on the initial directions of these streamlines. Shown in the second row of Figure 1 are examples of these curves. Together, these two coordinates provide a well-defined, orthogonal coordinate system.

Lemma 1 *For a given smooth surface S, the Darcyan coordinate system is well defined for all of S. It has the property that the level curves of the two coordinates intersect at right angles.*

Following Grenander et al. [13], we will call it the Darcyan coordinate system, named after the famous shape analyst D'Arcy Thompson.

Note that one can also develop this system by forming a polar coordinate system on the tangent space, $T_r(S) \equiv \mathbb{R}^2$, and by projecting level curves (circles and radial lines) onto S using the exponential maps (or geodesics along S) [5]. For this reason, these level curves of ξ_1 have also being called geodesic (polar) circles. Bronstein et al. [4] used largest geodesic circles to prune facial surfaces using multiple reference points; in that context, these circles were called geodesic masks. Some authors, e.g. [32], have used this system for matching features across faces but we are interested in a full shape analysis, that is, matching, deformation and comparisons of faces.

3 Elastic Matching/Deformation of Darcyan Level Curves

Once the Darcyan coordinate system is established, the task of elastically matching two surfaces is that of finding a mapping $\phi : S_1 \mapsto S_2$ that is optimal under the elastic metric. In our approach, ϕ can be decomposed into simpler components. There are two ways of decomposing $\phi(\xi_1, \xi_2)$: (i) into $\phi_1(\xi_1)\phi_2(\xi_2; \xi_1)$ and (ii) into $\phi_2(\xi_2)\phi_1(\xi_1; \xi_2)$. In the former decomposition, we first match the level curves of ξ_1, to find ϕ_1, and then for each pair of matched curves we find ϕ_2. This procedure is reversed for the latter decomposition. Similar to [36], we choose the former in this paper since the level curves of ξ_1 are more interesting and visually more appealing for shape analysis. We remark that the level curves of ξ_2 form facial profiles, similar to the idea used in [2] for face profiling.

We start by describing the estimation of ϕ_2. Let $C^1_{\xi_1}$ and $C^2_{\xi'_1}$ be two arbitrary level curves of the first coordinate in facial surfaces S_1 and S_2, respectively. Our goal is to find a mapping $\phi_2 : \xi_2 \mapsto \xi'_2$ that is a diffeomorphism and it minimizes the elastic distance (to be defined) between the two curves. We treat the two curves as closed, parameterized curves in $\mathbb{R}^3$ with fixed origins for parameterizations. Furthermore, we rescale them to be of the same length, say 2π. This viewpoint allows us to use one of many methods already available for an elastic analysis of closed curves. The key idea in elastic analysis is that the mapping ϕ_2 is nonlinear, i.e. points that are matched together are at unequal distances from their origins. Such a matching can be considered as an elastic matching, as one curve has to (locally) stretch, compress and bend to match the other. Several authors, starting with Younes [44], followed by Michor & Mumford [29], Mio et al. [31], and others have studied it for planar shapes. More recently Joshi et al. [18,17] have extended it to curves in $\mathbb{R}^n$ using an efficient representation of curves. Several authors, including Yezzi & Mennucci [43], have also used Riemannian metrics on spaces of curves but mainly to study evolution of curves rather than for shape analysis. Here we adopt the approach presented in Joshi et al. [18,17] because it greatly simplifies the elastic shape analysis. The main steps are: (i) define a space of closed curves of interest, (ii) impose a Riemannian structure on it using the elastic metric, and (iii) compute geodesic paths under this metric. These geodesic paths can then be interpreted as optimal elastic deformations of curves.

For the interval $I \equiv [0, 2\pi]$, let $\beta : I \to \mathbb{R}^3$ be a parameterized curve with a non-vanishing derivative everywhere. We represent its shape by the function $q : I \to \mathbb{R}^3$ defined as: $q(s) = \frac{\dot{\beta}(s)}{\sqrt{\|\dot{\beta}(s)\|}} \in \mathbb{R}^3$. Here, $\| \cdot \|$ is the standard Euclidean norm in $\mathbb{R}^3$. The quantity $\|q(s)\|$ is the square-root of the instantaneous speed on the curve β, whereas the ratio $\frac{q(s)}{\|q(s)\|}$ is the instantaneous direction at the point $s \in [0, 2\pi)$ along the curve. Let $\mathcal{Q} \equiv \{q = (q_1, q_2, q_3) | q(s) : I \to \mathbb{R}^3, \|q(s)\| \neq 0, \forall s\}$ be the space of all square integrable functions in $\mathbb{R}^3$. The closure condition for a curve β requires that $\int_I \dot{\beta}(s)ds = 0$, which translates to $\int_I \|q(s)\|q(s)\,ds = 0$. We define a mapping $\mathcal{G} : \mathcal{Q} \mapsto \mathbb{R}^4$, with components:

$$\mathcal{G}_1 = \int_I q_1(s)\,\|q(s)\|ds,\ \mathcal{G}_2 = \int_I q_2(s)\,\|q(s)\|ds,\ \mathcal{G}_3 = \int_I q_3(s)\,\|q(s)\|ds,\ \mathcal{G}_4 = \int_I \|q(s)\|^2 ds\ .$$

The space obtained by the inverse image $\mathcal{C} = \mathcal{G}^{-1}(0, 0, 0, 2\pi)$ is the space of all closed curves of length 2π, and this representation is invariant to translation and scaling. $\mathcal{C}$ is endowed with a Riemannian structure using the metric: for any two tangent vectors $v_1, v_2 \in T_q(\mathcal{C})$, we define

$$\langle v_1, v_2 \rangle = \int_I v_1(s)^T v_2(s)ds\ . \tag{1}$$

It is shown in [18] that this metric is equivalent to the elastic metric described in [31].

Next, we want a tool to compute geodesic paths between arbitrary elements of $\mathcal{C}$. There have been two prominent numerical approaches for computing geodesic paths on nonlinear manifolds. One approach uses the **shooting method** ([23,31]) where, given a pair of shapes, one finds a tangent direction at the first shape such that its image under the exponential map reaches as close to the second shape as possible. We will use another, more stable approach that uses **path-straightening** flows to find a geodesic between two shapes. In this approach, the given pair of shapes is connected by an initial arbitrary path that is iteratively "straightened" so as to minimize its length. The path-straightening method, proposed by Klassen et al [22], overcomes some of the practical limitations in the shooting method. Other authors, including Schmidt et al. [37] and Glaunes et al [10], have also presented other variational techniques for finding optimal matches.
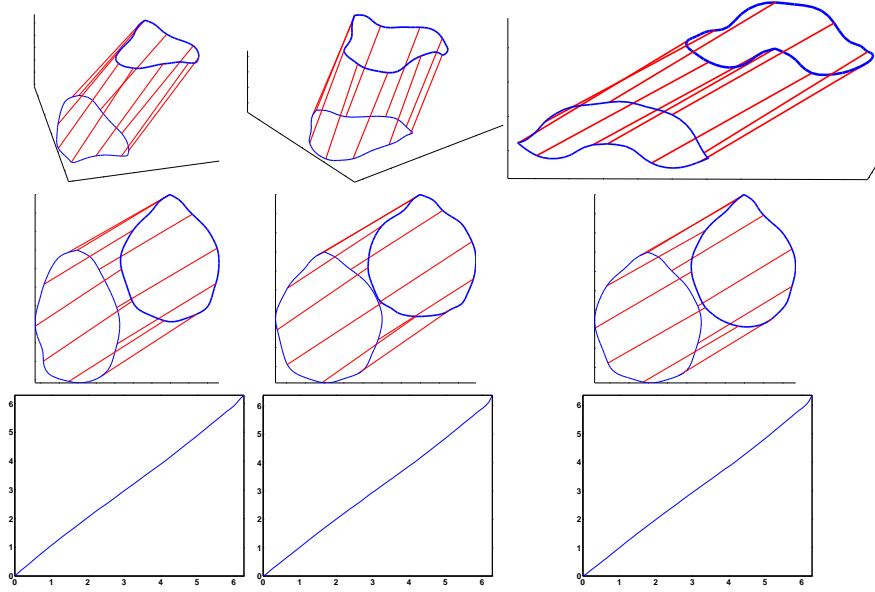
Fig. 2 Elastic matching of facial curves. The top and the middle rows show two different views of the matching points across the curves, while the bottom panels show optimal matching functions ϕ_2^*.

Given two curves, represented by q^0 and q^1, our goal is to find a geodesic between them in $\mathcal{C}$. Let $\alpha : [0, 1] \to \mathcal{C}$ be any path connecting q^0, q^1 in $\mathcal{C}$, i.e. $\alpha(0) = q^0$ and $\alpha(1) = q^1$. Then, the critical points of the energy

$$E[\alpha] = \frac{1}{2} \int_0^1 \langle \dot{\alpha}(t), \dot{\alpha}(t) \rangle \ dt \ , \tag{2}$$

with the inner product defined in Eqn. 1, are geodesics in $\mathcal{C}$ (this result is true on a general manifold [39]). As described by Klassen et al. [22] (for general shape manifolds), and Joshi et al. [18] (for this particular representation of curves), one can use a gradient approach to find a critical point of E and reach a geodesic. The distance between the two curves q^0 and q^1 is given by the length of the geodesic α:

$$d_c(q^0, q^1) = \int_0^1 (\langle \alpha'(t), \alpha'(t) \rangle)^{1/2} dt \ .$$

We will call this the **elastic** distance between the curves represented by q^0 and q^1.

Similar to Kendall's shape formulation [20], we define shapes of curves as invariant under rigid motions, uniform scaling and other such "shape-preserving" transformations. Since translation, scaling, and location of origin are already accounted for, we need to remove the variability associated with rotations and origin-preserving (positive) diffeomorphisms on I. In other words, the shape space for analyzing shapes of closed curves in $\mathbb{R}^3$ is given by $\mathcal{S} = \mathcal{C}/(SO(3) \times \mathcal{D})$, where $SO(3)$ is the three-dimensional rotation group and $\mathcal{D}$ is the set of origin-preserving positive diffeomorphisms on I. The task of finding a geodesic in $\mathcal{S}$ is that of finding a geodesic in $\mathcal{C}$ that achieves the distance:

$$d_s(q_1, q_2) = \min_{\gamma \in \mathcal{D}, O \in SO(3)} d_c(q_1, \gamma \cdot (Oq_2)) \ , \tag{3}$$

where $\gamma \cdot q_2$ is the re-parametrization of q_2 according to the formula: $\sqrt{\dot{\gamma}} q_2(\gamma)$. As described in [17], one can solve for this minimization on $\mathcal{D} \times SO(3)$ using a gradient approach; please refer to that paper for details. We will illustrate these ideas using some examples. Some examples of elastic matching of corresponding points across facial curves is shown in Figure 2; it shows three examples of matching pairs of facial curves and the corresponding optimal matching functions ϕ_2^*. The optimal γ, that minimizes the right side of Eqn. 3, provides the component ϕ_2^* of the optimal mapping ϕ^*. Figure 3 illustrates the idea of optimal elastic deformations between facial curves. It shows the geodesic paths between facial curves in $\mathcal{S}$ under the elastic
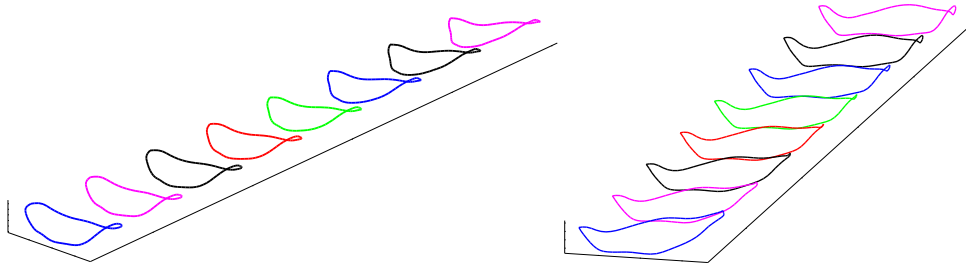
Fig. 3 Examples of optimal elastic deformations between facial curves extracted as level curves of ξ_1.

metric. In each case, the first and the last curve are the examples of facial curves and the intermediate curves are points along a geodesic path connecting them in $\mathcal{S}$.

4 Elastic Matching of Curves Across Facial Surfaces

Now that we have tools for elastically deforming arbitrary facial curves into each other, we consider the problem of matching curves across two faces. Each facial surface is denoted by an indexed collection of facial curves, $S_i \equiv \{C^i_{\xi_1} | \xi_1 \in [0, L]\}$, $i = 1, 2$. (We remind the reader that $C^i_{\xi_1}$ is the level curve of first Darcyan coordinate at the value ξ_1 for the facial surface S_i, $i = 1, 2$.) Our goal is to find ϕ_1 that optimally matches curves across these collections. We initially formulate the problem in the continuum, but will later discretize the domain to reach a computational solution. Let $\phi_1 : [0, L] \mapsto [0, L]$ be a diffeomorphic mapping that minimizes the cost function:

$$E_s[\phi_1; 0, L] = \int_0^L d_s(C^1_{\xi_1}, C^2_{\phi_1(\xi_1)})^2 d\xi_1 \;, \tag{4}$$

where d_s is as described in Eqn. 3. The solution requires a search over the infinite-dimensional space of all diffeomorphisms from $[0, L]$ to itself. There exists a computational approach to solving this type of problem in situations where the domain and the range of ϕ_1 are sampled evenly using a finite number of points. Termed as dynamic time warping, this technique has been used for matching points on curves by several authors, include Sebastian et al. [38] and Mio et al. [31]. For notational convenience we will rescale ξ_1 so that it takes values in $[0, 1]$, and we seek an optimal ϕ_1 on $\mathcal{D}_{[0,1]}$, the set of all diffeomorphisms from $[0, 1]$ to itself.

Let each facial surface be sampled evenly in ξ_1 at n points to give an indexed collection of n curves, each an element in $\mathcal{C}$. This results in an $n \times n$ graph, of the type shown in Figure 4, whose nodes denote possible matchings of corresponding curves from the two surfaces. The goal is now to find a piecewise linear $\phi_1 \in \mathcal{D}_{[0,1]}$, i.e. a curve that passes through the nodes on the graph, that minimizes the energy function given in Eqn. 4, and we do so using dynamic programming (DP). DP is useful in finding optimal paths between any two points on a finite graph when the cost associated with the path is additive over its segments. The piecewise-linear path between the points $(0, 0)$ to $(1, 1)$ is constrained to be such that: (i) the slope is always positive along the path, and (ii) it minimizes the elastic energy. To evaluate the elastic energy we will need the term $d_s(C^1_{(i/n)}, C^2_{(j/n)})$ for all i, j, as described in the previous section.

Consider a uniform, $n \times n$ grid in $\mathbb{R}^2$ such that the bottom left maps to $(0, 0)$ and the top right maps to indices $(1, 1)$ on this grid, as shown in Figure 4. For any point $(i/n, j/n)$, for $0 < i, j \le n$, let N_{ij} be the set of indices that are allowed to go to $(i/n, j/n)$. In principle, all the points $(k/n, l/n)$ such that $k < i$ and $l < j$ are valid, but one may restrict to a smaller set to seek a faster approximation. Let $L(k, l; i, j)$ denote a straight line joining the pairs $(k/n, l/n)$ and $(i/n, j/n)$. Then, the iterative optimization problem is:

$$(\hat{k}, \hat{l}) = \operatorname*{argmin}_{(k,l) \in N_{ij}} E_s[L(k, l; i, j); k/n, i/n] \;, \tag{5}$$

with E_s as defined in Eqn. 4. Define the minimum energy of reaching the point $(i/n, j/n)$, in an iterative fashion as:

$$H_s(i/n, j/n) = E_s[L(\hat{k}, \hat{l}; i, j), \hat{k}/n, i/n] + H_s(\hat{k}/n, \hat{l}/n) \;, \quad \text{with} \;\; H(0, 0) = 0 \;.$$
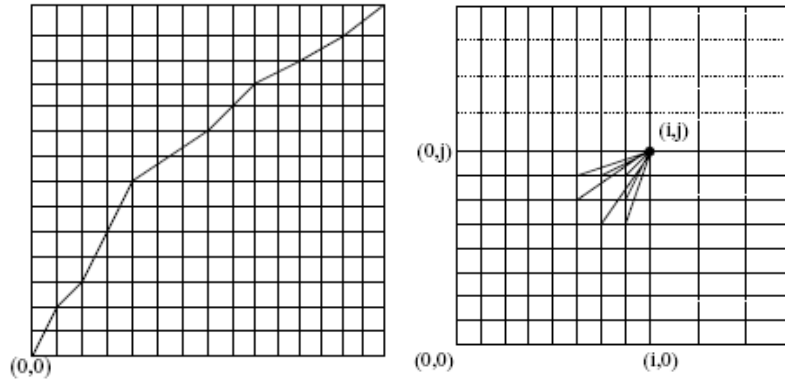
Fig. 4 Left: An example of a ϕ_1 function restricted to a finite graph. Right: An illustration of some nodes that are allowed to go to $(i/n, j/n)$ point on the graph.
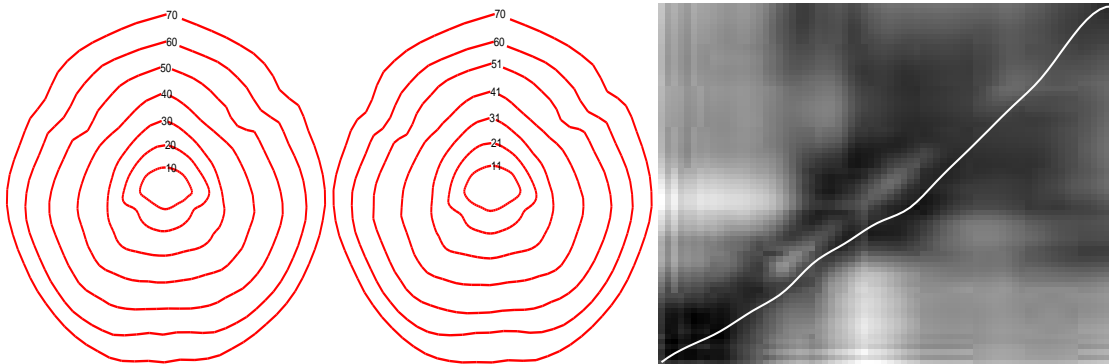

Fig. 5 The left two panels show a sub-sampled set of matched curves– 10 with 11,20 with 21, etc – across the two collections representing S_1 and S_2. The rightmost panel shows the optimal matching function ϕ_1^* between the two collections.

The optimization problem in Eqn. 5 is solved sequentially for each point $(i/n, j/n)$, starting from $(0, 0)$ and increasing i/n, j/n until the last point $(1, 1)$. Tracing the path that results in the optimal elastic energy $H_s(1, 1)$ provides a discrete version of the optimal ϕ_1. The set of nodes on the graph, contained in optimal ϕ_1, provide the matching pairs in the two collections. Shown in Figure 5 is an example of this idea for $n = 70$. The rightmost panel shows a pictorial view of the function H with darker shade corresponding to lower energy. Overlaid on this picture is the plot of optimal ϕ_1 for this case. Shown in the left two panels are the corresponding pairs of curves in the two faces. Only the matching curves, associated with the nodes that lie on optimal ϕ_1, are shown for clarity. According to this figure, the 10th curve in the first surface matches the 11th curve in the second, the 50th curve in the first surface matches the 51st curve in the second, and so on.

5 Elastic Deformation of Facial Surfaces

The previous two sections provide the solutions for solving optimal mappings ϕ_2^* (Section 3) and ϕ_1^* (Section 4). Together they provide an elastic matching of points across the two facial surfaces according to $\phi^*(\xi_1, \xi_2) = \phi_1^*(\xi_1)\phi_2^*(\xi_2; \xi_1)$.

The construction of geodesic paths between Darcyan curves in Section 3, using an elastic metric, allows us to realize optimal deformations between surfaces. Applying this idea repeatedly to each pair of registered curves gives us a full elastic deformation from one face to another. If we view each facial surface as an indexed collection of curves, indexed by ξ_1, this idea provides a path on the space of such collections, in going from one face to another. We will denote this path by Ω, where $\Omega(t)$, for any $t \in [0, 1]$ is an indexed collection of
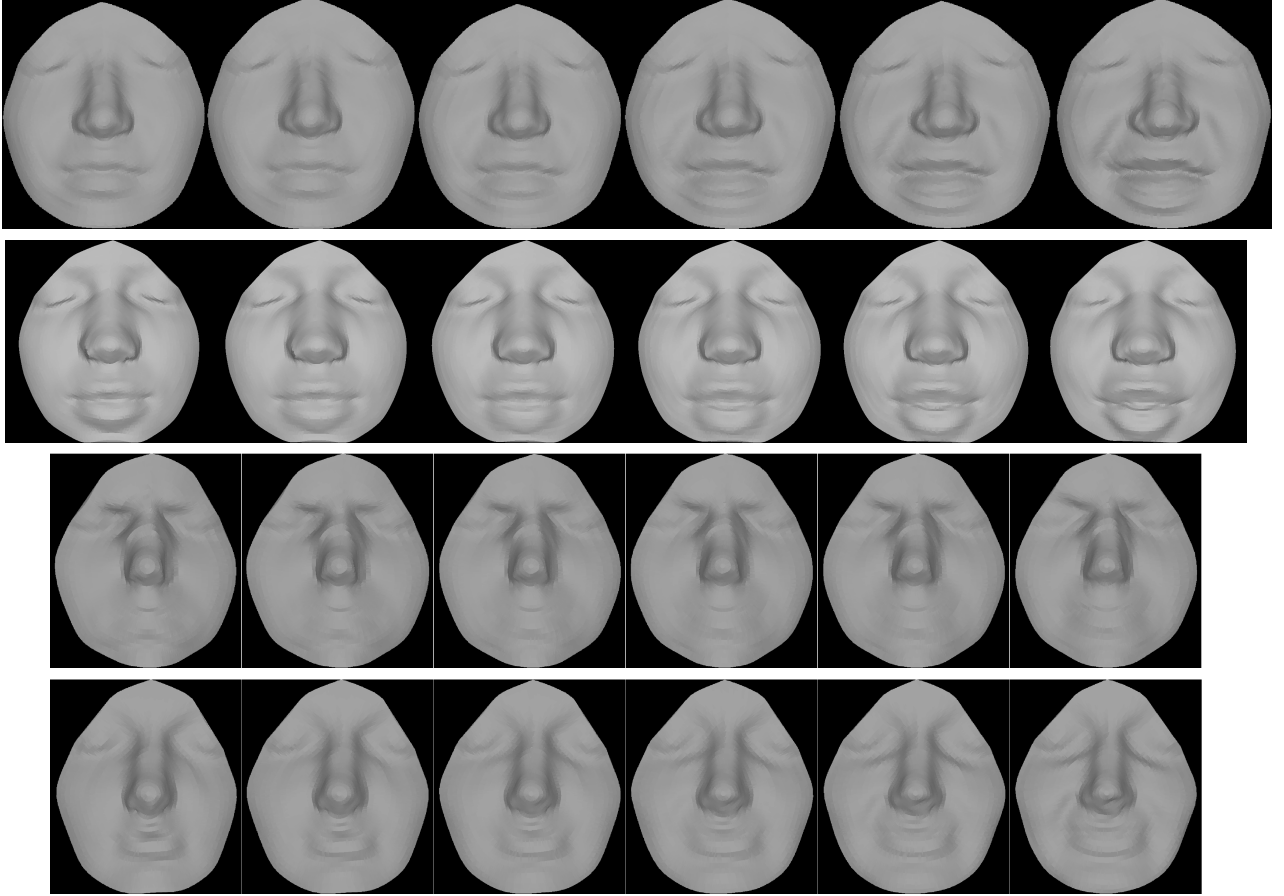
Fig. 6 Optimal deformations of facial surfaces into each other, both belonging to the same person, under the elastic energy.

points in $\mathcal{S}$. This path is optimal in the sense that it minimizes the total length:

$$d(S_1, S_2) \equiv \int_0^1 d_s(C^1_{\xi_1}, C^2_{\phi_1^*(\xi_1)}) d\xi_1 \ . \tag{6}$$

Shown in Figures 6 - 7 are examples of these deformations. Figure 6 shows four examples of deforming facial surfaces, one into another, when both the faces belong to the same person. Similarly, Figure 7 shows four examples of elastic deformations when the faces come from different persons.

Shown in Figure 8 is an example of comparison of deformations obtained under elastic and non-elastic metrics. The top row shows an example of non-elastic, or bending-only, deformation as described in Samir et al. [36], while the bottom row shows the elastic deformation introduced in this paper. Although the differences between the two deformations are not apparent visually, a closer inspection reveals a smoother deformation, especially around the lips, in the elastic case.

This solution also provides an elastic distance (Eqn. 6) between individual facial surfaces. To illustrate the usefulness of this distance, we show a matrix of pairwise distances between 12 faces in Figure 9, six of which belong to the same person while the remaining six belong to different persons. In this display, the distances have been scaled to take value between 0 and 1. As expected, the distances between faces of same person are much smaller compared to the distances between faces of different persons.

Since the Darcyan system relies critically on the location of the reference point (tip of the nose), and there is a possibility of numerical error in estimating this point, how stable is the framework presented here to such numerical errors? We have studied this issue experimentally and some results are presented in Figure 10. The left two panels show the level curves of ξ_1 on the same facial surface when the reference point is
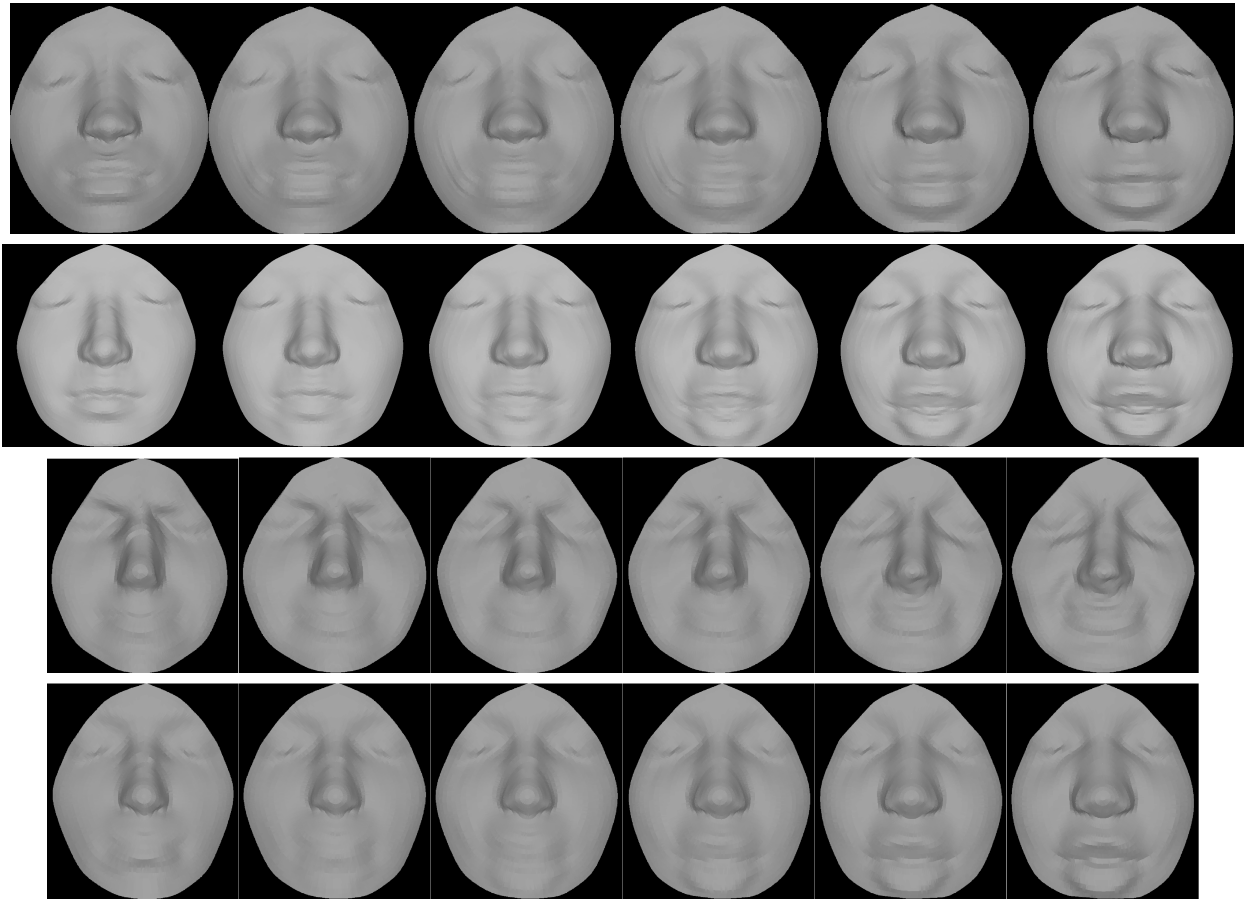
Fig. 7 Optimal deformations of a facial surface into another, belonging to a different person, under the elastic energy.
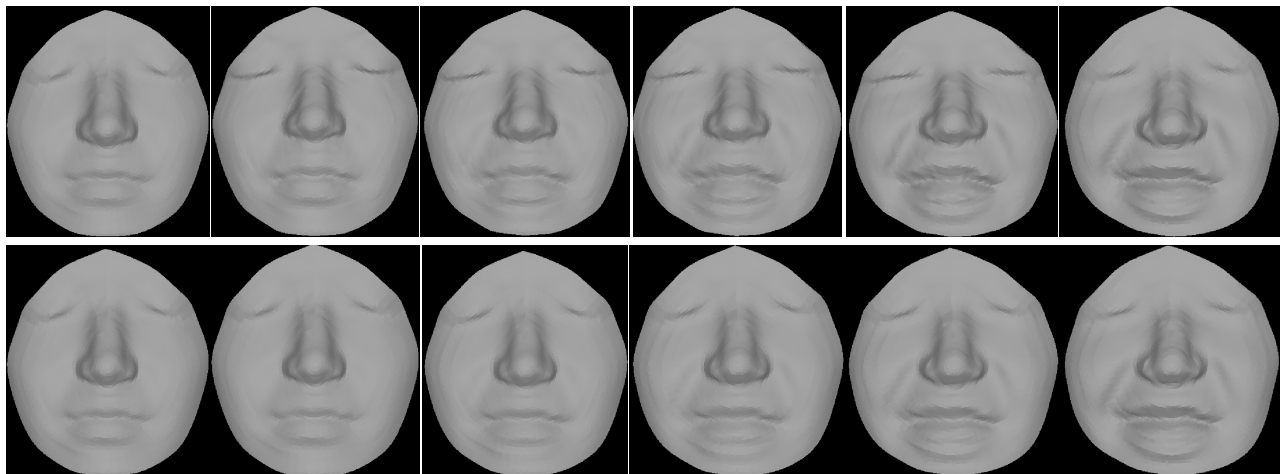

Fig. 8 Elastic versus non-elastic deformation. For the same pair of faces we show geodesic paths (deformations) using the non-elastic metric (top, as described in Samir et al. [36]) and the elastic metric (bottom) used in this paper.
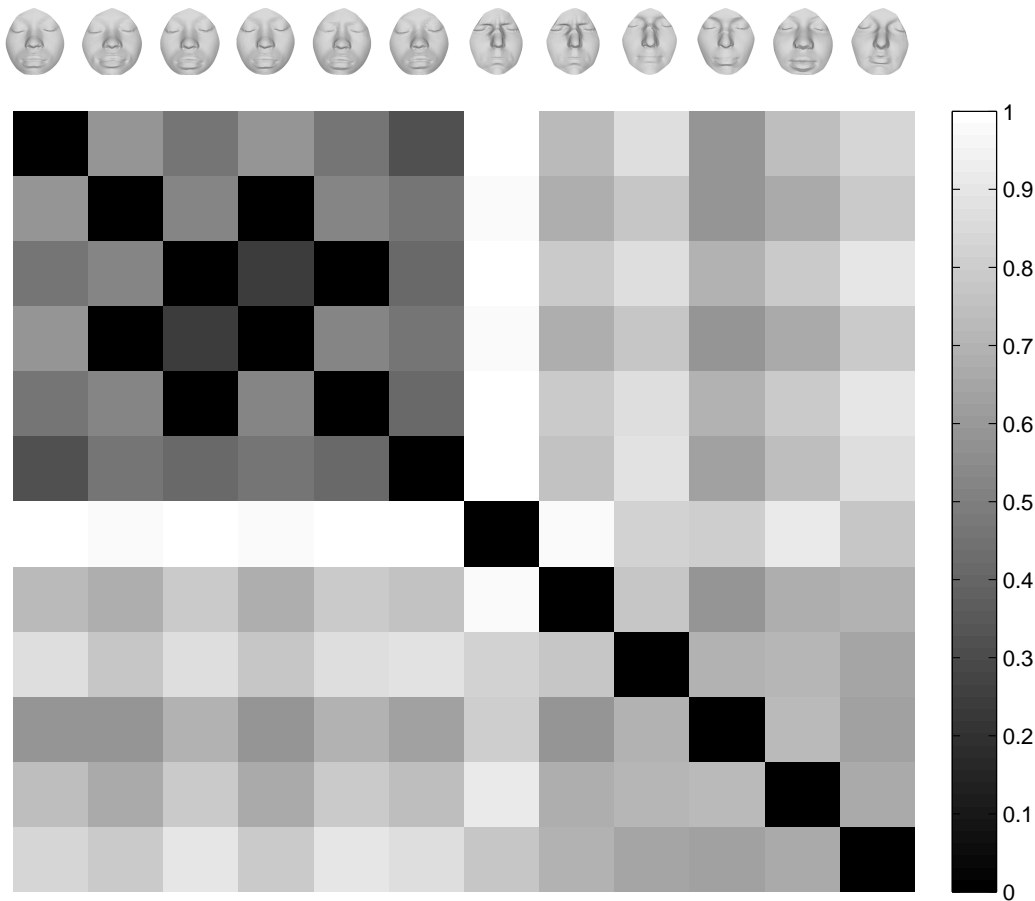
Fig. 9 Matrix of (scaled) pairwise distances between 12 facial surfaces. The first six faces belong to the same person while the remaining six are all different individuals.
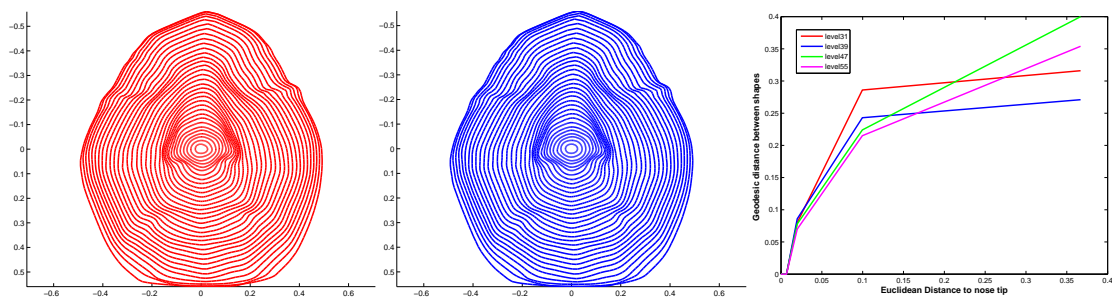

Fig. 10 Left two panels show level curves of Darcyan coordinate ξ_1 when the location of reference point is changed slightly. The panels shows the variation in shapes of level curves versus changes in locations of reference points, for curves at four levels of ξ_1.

located in a slightly different spot. The right plot shows the variation in shapes of levels curves, at four different levels, versus changes in the location of the reference point. For example, one curve in this plot shows the change in shapes of level curve 31, measured using the geodesic distance d_s, when the reference point is moved away from the actual tip of the nose. These variations are found to be sub-linear, implying a stability of shape analysis with respect to reference point location.

6 Summary

We have presented a Riemannian framework for analyzing shapes of facial surfaces for the purposes of matching, comparing, and deforming them. This approach uses a Darcyan coordinate system, a system that is intrinsic to a facial surface. This system is centered at the tip of the nose and its two coordinates ξ_1 and ξ_2 specify distances from this tip and the plane of symmetry, respectively. In this coordinate system, the task of elastic matching and deforming is broken into two distinct components. One component matches level curves of each surfaces, and the other component matches points along each pair of matched curves. These tasks are performed using geodesics on appropriate manifolds of curves and surfaces, with Riemannian metrics that measure elastic deformation. In other words, geodesics provide the optimal deformations of curves and surfaces across faces.

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